Combinatorial canonical form (CCF) of a layered mixed (LM-)matrix

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Layered mixed (LM-)matrix and their combinatorial canonical form (CCF) are are useful tools in the structural approach to systems analysis by means of matroid. Here is a very brief mathematical description.

Let K be a subfield of a field F. A matrix A over F is called a layered mixed matrix (or an LM-matrix) with respect to K if it takes the following form (possibly after a permutation of rows): $A = \begin{pmatrix} Q \\ T \end{pmatrix}$, where Q and T meet the following requirements:

(i) $Q = (Q_{ij})$ is a matrix over K, and

(ii) $T = (T_{ij})$ is a matrix over F such that the set of its nonzero entries is algebraically independent over K.

With an LM-matrix A we associate a submodular function p defined on its column set C by

$$p(J) = \rho(J) + \gamma(J) - |J|,$$

where

$$\rho(J) = \mathrm{rank} Q[R_Q, J], \quad \gamma(J) = |\bigcup_{j \in J} \{i \in R_T \mid T_{ij} \neq 0\}|, \quad J \subseteq C.$$

The rank of A is characterized by the minimum of p:

$$\mathrm{rank}\ A = \min\{p(J) \mid J \subseteq C\} + |C|.$$

This is an extension of the well-known min-max characterization of the term-rank of a matrix or the maximum matching in a bipartite graph, which is ascribed to J. Egerváry, D. König, P. Hall, R. Rado, O. Ore and others.

The admissible transformation for an LM-matrix A of means a transformation of the form:

$$P_{\mathrm{r}} \begin{pmatrix} S & O \\ O & I \end{pmatrix} \begin{pmatrix} Q \\ T \end{pmatrix} P_{\mathrm{c}},$$

where S is a nonsingular matrix over the subfield K, and P_{r} and P_{c} are permutation matrices. The admissible transformation brings an LM-matrix into another LM-matrix and two LM-matrices are said to be LM-equivalent if they are connected by an admissible transformation. The function p is an invariant under the LM-equivalence.

There exists a finest block-triangular matrix, called the combinatorial canonical form (or CCF for short), among the matrices which are LM-equivalent to each other. This can be shown based on the submodular function p with the aid of a general principle, the Jordan-Hölder type decomposition principle, for a submodular function.

In the very special case with Q being empty, the admissible transformation reduces to mere permutations of rows and columns, and then the CCF reduces to the canonical decomposition of a bipartite graph due to Dulmage and Mendelsohn.

In the other extreme case with T being empty, the CCF is essentially the Gaussian elimination. Symbolically,

$$\text{CCF} = \text{DM-decomposition} + \text{LU-decomposition}.$$

The CCF can be computed efficiently (in polynomial time) using the matroid partition algorithm. The indecomposability under the admissible transformation defines the notion of LM-irreducibility, an extension of the well-studied notion of full irreducibility.

LM-equivalence has a natural physical meaning. For example, an electrical network is typically described by means of an LM-matrix when currents in and voltages across branches are chosen as the elementary variables. In that case, the T-part represents the constitutive equations (Ohm's law, etc.) and the Q-part the structural equations for Kirchhoff's current and voltage laws. As is well known there are a number of different ways of expressing the conservation laws. The LM-equivalence accounts exactly for the degree of freedom in expressing Kirchhoff's laws.

References

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